Applications in Probability

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- Consider a random variable X that takes values in an interval or a union of intervals of real numbers. We denote the set of possible values of X by S, and call X a *continuous random variable*. Suppose that $f(x)$ is an nonnegative function defined on S such that
  - $\int_S f(x)\,dx = 1$.
  - For any subset $A \subset S$, the probability of $X \in A$ is $P(X \in A) = \int_A f(x)\,dx$.

  Then $f(x)$ is called the *probability density function* (p.d.f.) of X. We often let $f(x)$ be 0 for x not in S so that $f(x)$ is defined on R.

- For a continuous random variable X with probability density function $f(x)$, we define the *expected value* ($E(X)$), *variance* (var(X)), and *standard deviation* ($\sigma(X)$)of X as

$$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx\ ; \qquad \mathrm{var}(X) = \int_{-\infty}^{\infty} (x - E(X))^2 f(x)\,dx\ ; \qquad \sigma(X) = \sqrt{\mathrm{var}(X)}.$$

  As you can see, definitions of these values involve improper integrals!

- For a random variable X with p.d.f. $f(x)$, we define the *distribution function* of X, $F(x)$, as

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt.$$

  By the Fundamental Theorem of Calculus, we know that $F'(x) = f(x)$. This provides us a way to find the probability density function. For example, if $Y = aX + b$ where $a > 0$, then the distribution function of Y is

$$F(y) = P(Y \le y) = P(aX + b \le y) = P(X \le \frac{y-b}{a}) = \int_{-\infty}^{\frac{y-b}{a}} f(x)dx.$$

  And the p.d.f of Y is $\dfrac{d}{dy}F(y) = \dfrac{d}{dy}\displaystyle\int_{-\infty}^{\frac{y-b}{a}} f(x)dx = \frac{1}{a}\,f(\frac{y-b}{a})$.

1. Suppose that random variable X has probability density function $f_X(x) = \dfrac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, where μ, σ are constants and $\sigma > 0$.

   (a) Given $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$, show that $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = 1$.

   (b) Compute the expected value and standard deviation of X.

   (c) Define $Y = aX + b$ where a, b are constants and $a > 0$. Find the probability density function of Y, $f(y)$. What is the expected value and standard deviation of Y?

   (d) Find constants c and d such that $Z = cX + d$ has expected value 0 and standard deviation 1.(i.e. Z has the standard normal distribution.)

$\boxed{1.\,a)}$ $$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx + \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx .$$

And $$\frac{1}{\sqrt{2\pi}\sigma}\int_0^t e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \underset{\begin{cases} u=\frac{x-\mu}{\sqrt{2}\sigma} \\ du=\frac{1}{\sqrt{2}\sigma}dx \end{cases}}{=\!=} \frac{1}{\sqrt{\pi}}\int_0^{\frac{t-\mu}{\sqrt{2}\sigma}} e^{-u^2} du \xrightarrow[\text{as } t\to\infty]{} \frac{1}{\sqrt{\pi}}\int_0^{\infty} e^{-u^2} du = \frac{1}{2}$$

Similarly $$\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{2}.$$

Hence $$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1.$$

$\boxed{1\,b)}$ $$E(X) = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \int_{-\infty}^{\mu} \frac{x}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$+ \int_{\mu}^{\infty} \frac{x}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx .$$

Moreover, $$\int_{\mu}^{t} \frac{x}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \underset{du=-\frac{1}{\sqrt{2}\sigma}dx}{\overset{u=\frac{x-\mu}{\sqrt{2}\sigma}}{=\!=}} \frac{1}{\sqrt{\pi}}\int_0^{\frac{t-\mu}{\sqrt{2}\sigma}} (\sqrt{2}\sigma u+\mu)\, e^{-u^2} du$$

$$= \frac{-\sqrt{2}\sigma}{\sqrt{\pi}\,2} e^{-u^2}\Big|_0^{\frac{t-\mu}{\sqrt{2}\sigma}} + \frac{\mu}{\sqrt{\pi}}\int_0^{\frac{t-\mu}{\sqrt{2}\sigma}} e^{-u^2} du$$

$$\xrightarrow[\text{as } t\to\infty]{} \frac{\sigma}{\sqrt{2\pi}} + \frac{\mu}{\sqrt{\pi}}\int_0^{\infty} e^{-u^2} du = \frac{\sigma}{\sqrt{2\pi}} + \frac{\mu}{2}.$$

Similarly $\int_{-\infty}^{\mu} \frac{x}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{\sigma^2}} dx = -\frac{\sigma}{\sqrt{2\pi}} + \frac{\mu}{2}$.

Hence $E(x) = \left(\frac{\sigma}{\sqrt{2\pi}} + \frac{\mu}{2}\right) + \left(-\frac{\sigma}{\sqrt{2\pi}} + \frac{\mu}{2}\right) = \mu$.

$$\mathrm{Var}(X) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \left[\int_{-\infty}^{\mu} (x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx + \int_{\mu}^{\infty} (x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \right].$$

And $\frac{1}{\sqrt{2\pi}\sigma} \int_{\mu}^{t} (x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \underset{u=\frac{x-\mu}{\sqrt{2}\sigma}}{=\!=} \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{t-\mu}{\sqrt{2}\sigma}} 2\sigma^2 u^2 e^{-u^2} du$

$$= \frac{1}{\sqrt{\pi}} \sigma^2 \left[u(-e^{-u^2}) \Big|_{u=0}^{u=\frac{t-\mu}{\sqrt{2}\sigma}} + \int_{0}^{\frac{t-\mu}{\sqrt{2}\sigma}} e^{-u^2} du \right]$$

$$\xrightarrow[\text{as } t\to\infty]{} \frac{\sigma^2}{\sqrt{\pi}} \int_0^{\infty} e^{-u^2} du = \frac{\sigma^2}{2}.$$

Similarly, $\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\mu} (x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{\sigma^2}{2}$.

Hence $\mathrm{var}(X) = \frac{\sigma^2}{2} + \frac{\sigma^2}{2} = \sigma^2$. And $\sigma(X) = \sqrt{\sigma^2} = \sigma$.

$\boxed{1c)}$ The distribution function of Y is

$F(y) = P_r(Y \leq y) = P_r(aX+b \leq y) = P_r\left(X \leq \frac{y-b}{a}\right)$

$= \int_{-\infty}^{\frac{y-b}{a}} f_X(x)\,dx$. The probability density function of

Y is $f_Y(y) = \frac{d}{dy}F(y) = f_X\left(\frac{y-b}{a}\right) \times \frac{1}{a}$

$$= \frac{1}{\sqrt{2\pi}\,\sigma a} e^{-\frac{\left(\frac{y-b}{a}-\mu\right)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}\,\sigma a} e^{-\frac{(y-b-a\mu)^2}{2a^2\sigma^2}}.$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_y} e^{-\frac{(y-y_\mu)^2}{2\sigma_y^2}},$$ where $y_\mu = a\mu+b$, $y_\sigma = a\cdot\sigma$.

Hence by part b), we know that the expected value of Y is $a\mu+b$, and the standard deviation of Y is $a\sigma$.

$\boxed{1d)}$ The expected value of $Z = cX+d$ is $c\mu+d$ and the standard deviation is $c\cdot\sigma$. We want $c\mu+d=0$ and

$c\cdot\sigma = 1$. Hence $c = \frac{1}{\sigma}$, $d = -\frac{\mu}{\sigma}$.

2. It is said that Mr. K is very smart with IQ 157. Let us investigate what the scores mean.

   (a) Find information regarding IQ online. What is its probability density function? What are the expected value and standard deviation of IQ?

   (b) Find constants a and b such that a IQ + b is the standard normal distribution.

   (c) Now estimate $\Pr(IQ \geq 157)$. First, find c such that $\Pr(IQ \geq 157) = \Pr(X \geq c)$ where X has the standard normal distribution. Then, look up the corresponding probability in a table.

a) $IQ \sim \frac{1}{\sqrt{2\pi}\cdot 15} e^{-\frac{(x-100)^2}{2\times 15^2}}$. The expected value is 100, and the standard deviation is 15.

b) $aIQ+b$ has expected value $a\times 100+b$ and standard deviation $a\cdot 15$. Let $a=\frac{1}{15}$, $b=-\frac{100}{15}$. Then $\frac{IQ-100}{15}$ is the standard normal distribution.

c) $\Pr(IQ \geq 157) = \Pr\left(\frac{IQ-100}{15} \geq \frac{157-100}{15}\right) = \Pr\left(X \geq \frac{19}{5}\right)$

≈ 0.00007

3. Suppose that X is a random variable with standard normal distribution. Let $Y = X^2$.

   (a) Write the distribution function of Y as an integral. (You don't need to integrate it.)

   (b) Find the probability density function of Y. (This probability density function is called the *Chi-Square Distribution.*)

   (c) Compute the expected value and variance of Y.

a) The distribution function of Y is $F(y) = \Pr(Y \leq y)$

$$= \Pr(X^2 \leq y) = \Pr(-\sqrt{y} \leq X \leq \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

for $y \geq 0$ and $F(y) = 0$ for $y < 0$.

b) The probability density function of Y is

$$f(y) = \frac{d}{dy} F(y) = \frac{d}{dy} \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \left(e^{-\frac{y}{2}} \cdot \frac{1}{2\sqrt{y}} + e^{-\frac{y}{2}} \frac{1}{2\sqrt{y}} \right)$$

$$= \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}}$$ for $y > 0$ and $f(y) = 0$ for $y \leq 0$.

(c) $E(Y) = \int_0^\infty y f(y)\,dy = \int_0^\infty \sqrt{\frac{y}{2\pi}}\, e^{-\frac{y}{2}}\,dy$

$= \frac{1}{\sqrt{\pi}} \lim_{t\to\infty} \int_0^t \sqrt{\frac{y}{2}}\, e^{-\frac{y}{2}}\,dy \underset{\text{let } u=\sqrt{\frac{y}{2}}}{=} \frac{1}{\sqrt{\pi}} \lim_{t\to\infty} \int_0^{\sqrt{\frac{t}{2}}} 4u^2 e^{-u^2}\,du.$

$4u\,du = dy$

$= \frac{2}{\sqrt{\pi}} \lim_{t\to\infty} \left[-u\cdot e^{-u^2} \Big|_0^{\sqrt{\frac{t}{2}}} + \int_0^{\sqrt{\frac{t}{2}}} e^{-u^2}\,du \right]$

$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2}\,du = 1.$

$\operatorname{var}(Y) = \int_0^\infty y^2 f(y)\,dy - (E(Y))^2 = \frac{1}{\sqrt{2\pi}} \int_0^\infty y^{\frac{3}{2}} e^{-\frac{y}{2}}\,dy - 1.$

And $\int_0^\infty y^{\frac{3}{2}} e^{-\frac{y}{2}}\,dy = \lim_{t\to\infty} \int_0^t y^{\frac{3}{2}} e^{-\frac{y}{2}}\,dy = \lim_{t\to\infty} \Big[-2y^{\frac{3}{2}} e^{-\frac{y}{2}} \Big|_0^t$

$+ 3\int_0^t y^{\frac{1}{2}} e^{-\frac{y}{2}}\,dy \Big] = 3\int_0^\infty \sqrt{y}\cdot e^{-\frac{y}{2}}\,dy = 3\cdot\sqrt{2\pi}$

Hence $\operatorname{var}(Y) = \frac{1}{\sqrt{2\pi}}\cdot 3\sqrt{2\pi} - 1 = 2.$

$\sigma(Y) = \sqrt{\operatorname{var}(Y)} = \sqrt{2}.$